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Exact Analytical Solution to Equations of Perihelion Advance in General Relativity

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Abstract Previously the perihelion advance in binary system was computed approximately. We will present an exact analytical solution to nonlinear differential equation of perihelion advance by method of Jacobian elliptic function and the advanced angle between successive perihelions.

Keywords Perihelion advance · Jacobian elliptic function · Exact solution

1 Introduction

Perihelion advance is an old problem in astrophysics. In the literatures [1–3] perihelion advance in binary star system was computed approximately. This approximation method is valid if $2GM/rc^2 \ll 1$. If the condition $2GM/rc^2 \ll 1$ is not satisfied, the approximation method is not valid. We shall give the first exact analytical solution in the world to perihelion advance equation by method of Jacobian elliptic function and the advanced angle between successive perihelions.

2 General Particle Motion [4]

The paths of particles with mass moving in the vicinity of a spherical massive objective M are given by the timelike geodesics of spacetime. For a timelike geodesics we may use its

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proper time τ as an affine parameter. The four geodesic equations are given by

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^{\mu}}\right)-\frac{\partial L}{\partial x^{\mu}}=0, \tag{1}$$

where

$$\begin{aligned} L(\dot{x}^{\sigma}, x^{\sigma}) &\equiv \frac{1}{2}g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} \\ &= \frac{1}{2}[c^2(1-2m/r)\dot{t}^2-(1-2m/r)^{-1}\dot{r}^2-r^2(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)]. \end{aligned} \tag{2}$$

Here dots denote derivatives with respect to τ, the coordinates are $x^0 \equiv t$, $x^1 \equiv r$, $x^2 \equiv \theta$, $x^3 \equiv \phi$, and we have put $m = GM/c^2$.

Because of the spherical symmetry, there is no loss of generality in confining our attention to particles moving in the "equatorial plane" given by $\theta = \frac{\pi}{2}$. With this value for θ, the third ($\mu = 2$) of (1) is satisfied, and the second of these ($\mu = 1$) reduces to

$$\left(1-\frac{2m}{r}\right)^{-1}\ddot{r}+\frac{mc^2}{r^2}\dot{t}^2-\left(1-\frac{2m}{r}\right)^{-2}\frac{m}{r^2}\dot{r}^2-r\dot{\phi}^2=0. \tag{3}$$

Since t and ϕ are cyclic coordinates, we have immediate integrals of the two remaining equations: $\partial L/\partial\dot{t} = \text{const}$, $\partial L/\partial\dot{\phi} = \text{const}$. With $\theta = \pi/2$ these are:

$$(1-2m/r)\dot{t}=k, \tag{4}$$

$$r^2\dot{\phi}=h, \tag{5}$$

where k and h are integration constants. We also have the following relation

$$c^2d\tau^2 \equiv g_{\mu\nu}dx^{\mu}dx^{\nu}$$

which defines τ. With $\theta = \pi/2$ this becomes

$$c^2(1-2m/r)\dot{t}^2-(1-2m/r)^{-1}\dot{r}^2-r^2\dot{\phi}^2=c^2, \tag{6}$$

and may often be used in place of the rather complicated equation (3).

Equation (4) gives the relation between the coordinate time t and the proper time τ; (5) is analogous to the equation of conservation of angular momentum; as we shall see, (6) yields an equation analogous to that expressing the conservation energy.

Equation (6) gives

$$c^2(1-2m/r)\dot{t}^2/\dot{\phi}^2-(1-2m/r)^{-1}(dr/d\phi)^2-r^2=c^2/\dot{\phi}^2, \tag{7}$$

and substituting for $\dot{\phi}$ and $\dot{t}$ from (4) and (5) gives

$$(dr/d\phi)^2+r^2(1+c^2r^2/h^2)(1-2m/r)-c^2k^2r^4/h^2=0. \tag{8}$$

If we put $u \equiv 1/r$ and $m = GM/c^2$ this reduces to

$$\left(\frac{du}{d\phi}\right)^2+u^2=E+\frac{2GM}{h^2}u+\frac{2GM}{c^2}u^3, \tag{9}$$

where $E \equiv c^2(k^2-1)/h^2$. Comparing this with the Newtonian equation (10) in the next section, we see that it corresponds to an energy equation. Comparison also shows that the last term on the right is nonlinear. So (9) is nonlinear and very difficult to obtain exact analytical solution. We shall give the first exact analytical solution in the world to (9) by method of Jacobian elliptic function in the next section and compare it with observation values. In theory (9) may be integrated to give u, and hence r, as a function of ϕ, to obtain the particle paths in the "equatorial plane".

3 Exact Analytical Solution to Perihelion Advance Equation

For a particle moving in the equatorial plane under the Newtonian gravitational attraction of a spherical object of mass M situated at the origin, classical angular momentum and energy considerations lead to the equation

$$(du/d\phi)^2 + u^2 = E + 2GMu/h^2, \tag{10}$$

where $u \equiv 1/r$, E is a constant related to the energy of the orbit, and h is the angular momentum per unit mass given by $r^2 d\phi/dt = h$ [4]. The solution of this equation is well known from mechanics as

$$u = (GM/h^2)[1 + e\cos(\phi - \phi_0)], \tag{11}$$

where ϕ_0 is a constant of integration, and $e \equiv 1 + Eh^4/G^2m^2$ [4]. Equation (11) is that of a conic section with eccentricity e.

The general-relativistic analogue of (10) is (9), and we expect the extra term (equal to $2GMu^3/c^2$) to perturb the Newtonian orbit in some way. If we take the Schwarzschild solution as a model for the solar system, treating the planets as particles, then this extra term makes its presence felt by an advance of the perihelion (i.e. the point of closest approach to the sun) in each circuit of a planet about the Sun.

Let us denote $2GM/c^2$ by ε. Equation (9) becomes

$$(du/d\phi)^2 = \varepsilon u^3 - u^2 + \frac{2GM}{h^2}u + E. \tag{12}$$

Aphelion and perihelion occur where $du/d\phi = 0$, i.e. at values of u satisfying

$$\varepsilon u^3 - u^2 + (2GM/h^2)u + E = 0. \tag{13}$$

This is a cubic equation for u. Let us make transformation

$$u = y + 1/3\varepsilon. \tag{14}$$

Then (13) becomes

$$y^3 + py + s = 0, \tag{15}$$

where

$$p = \frac{2GM}{\varepsilon h^2} - \frac{1}{3\varepsilon^2},$$
$$s = -\frac{2}{27\varepsilon^3} + \frac{2GM}{3\varepsilon^2 h^2} + \frac{E}{\varepsilon}.$$

From physical meaning of u, we know that roots of (13) should be real. Therefore roots of (15) should also be real and can be solved as

$$
\begin{aligned}
y_1 &= \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right),\\
y_2 &= \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right),\\
y_3 &= \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\frac{\theta}{3},
\end{aligned}
\tag{16}
$$

where $\theta = \arccos[(-\frac{s}{2})(-\frac{p}{3})^{-\frac{3}{2}}]$. It is very easy to verify that [see Appendix] $y_1 \le y_2 \le y_3$. By putting y_1, y_2, y_3 of (16) into (14), we can obtain solutions of (13) as follows

$$
\begin{aligned}
u_1 &= \frac{1}{3\varepsilon} + \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\left(\theta + \frac{2\pi}{3}\right),\\
u_2 &= \frac{1}{3\varepsilon} + \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\left(\theta + \frac{4\pi}{3}\right),\\
u_3 &= \frac{1}{3\varepsilon} + \left(-\frac{4p}{3}\right)^{\frac{1}{2}} \cos\theta.
\end{aligned}
\tag{17}
$$

It is very easy to see that $u_1 \le u_2 \le u_3$. Because $(\frac{du}{d\phi})^2 \ge 0$, physical-significant field of the solutions of (12) is shown as $u \in [u_1, u_2]$ or $u \in [u_3, \infty]$. When $u \in [u_3, \infty]$,the planet falls into Sun; when $u \in [u_1, u_2]$, the planet orbits in the range that u_1 is aphelion and u_2 is perihelion. Equation (12) becomes

$$
\left(\frac{du}{d\phi}\right)^2 = \varepsilon(u - u_1)(u - u_2)(u - u_3). \tag{18}
$$

Equation (18) can be integrated as

$$
\int_{\phi_0}^{\phi} d\phi = \int_{u_1}^{u} \frac{du}{[\varepsilon(u - u_1)(u - u_2)(u - u_3)]^{\frac{1}{2}}}, \tag{19}
$$

For simplicity, we require that

$$
u = u_1 + (u_2 - u_1)q^2, \tag{20}
$$

where $q \in [0, 1]$. Then (19) becomes

$$
\begin{aligned}
\int_{\phi_0}^{\phi} d\phi &= \frac{2}{\sqrt{\varepsilon}} \int_0^q \frac{dq}{\sqrt{u_3 - u_1}[(1 - q^2)(1 - \frac{u_2 - u_1}{u_3 - u_1} q^2)]^{\frac{1}{2}}}\\
&= \frac{2}{\sqrt{\varepsilon(u_3 - u_1)}} \int_0^q \frac{dq}{[(1 - q^2)(1 - k^2 q^2)]^{\frac{1}{2}}}.
\end{aligned}
\tag{21}
$$

It is well known that the solution of (21) is expressed by Jacobian elliptic function [5] as

$$q = \mathrm{sn}\left[\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi - \phi_0), k\right], \tag{22}$$

where

$$k = \sqrt{\frac{u_2 - u_1}{u_3 - u_1}}. \tag{23}$$

By using (22), (20) can be written as

$$\begin{aligned} u &= u_1 + (u_2 - u_1)\,\mathrm{sn}^2\left[\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi - \phi_0), k\right], \\ &= u_2 - (u_2 - u_1)\,\mathrm{cn}^2\left[\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi - \phi_0), k\right]. \end{aligned} \tag{24}$$

This is the exact analytical solution of (12) or (9).

4 The Advanced Angle between Successive Perihelions

When $u = u_1$, (24) gives

$$\mathrm{sn}^2\left[\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi - \phi_0), k\right] = 0. \tag{25}$$

When $u = u_2$, (24) gives

$$\mathrm{cn}^2\left[\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi - \phi_0), k\right] = 0. \tag{26}$$

We can obtain the zeros of (25) and (26) [5] to find the angle $\Delta\phi = \phi_2 - \phi_1$ between an aphelion and the next perihelion. The zeros of (25) are

$$\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi_1 - \phi_0) = 2mK + 2nK'i. \tag{27}$$

The zeros of (26) are

$$\sqrt{\frac{\varepsilon(u_3 - u_1)}{4}}(\phi_2 - \phi_0) = (2m + 1)K + 2nK'i, \tag{28}$$

$$\Delta\phi = \phi_2 - \phi_1 = \frac{2K}{\sqrt{\varepsilon(u_3 - u_1)}}, \tag{29}$$

where [5]

$$K = \frac{\pi}{2}\left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots + \left[\frac{(2n-1)!!}{n!2^n}\right]^2 k^{2n} + \cdots\right\}. \tag{30}$$

Doubling $\Delta\phi$ gives the angle between successive perihelions, and shows that in each circuit this is advanced by

$$\begin{aligned}2\Delta\phi - 2\pi &= \frac{2\pi}{\sqrt{\varepsilon(u_3-u_1)}}\left\{1+\left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2 k^4 + \cdots\right.\\ &\quad \left. + \left[\frac{(2n-1)!!}{n!2^n}\right]^2 k^{2n} + \cdots\right\} - 2\pi\\ &= \frac{2\pi}{\sqrt{1-(2u_1+u_2)\varepsilon}}\left\{1+\left(\frac{1}{2}\right)^2 \frac{(u_2-u_1)\varepsilon}{1-(2u_1+u_2)\varepsilon}\right.\\ &\quad + \left(\frac{1\cdot 3}{2\cdot 4}\right)^2\left[\frac{(u_2-u_1)\varepsilon}{1-(2u_1+u_2)\varepsilon}\right]^2 + \cdots\\ &\quad \left. + \left[\frac{(2n-1)!!}{n!2^n}\right]^2\left[\frac{(u_2-u_1)\varepsilon}{1-(2u_1+u_2)\varepsilon}\right]^n + \cdots\right\} - 2\pi, \end{aligned} \tag{31}$$

on using (23) and $u_1+u_2+u_3 = \frac{1}{\varepsilon}$ from (13). This gives the advanced angle in each circuit by exact analytical method. Equation (31) may be applied in binary star system.

5 Application for the Planet Mercury

The dimension of $\varepsilon = 2GM/c^2$ is of length, and is small when compared with values of r corresponding to planetary orbits. To first order in ε, (31) becomes

$$2\Delta\phi - 2\pi = \frac{3\pi}{2}(u_1+u_2)\varepsilon = \frac{3GM\pi}{c^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right), \tag{32}$$

where r_1 and r_2 are the values of r at aphelion and perihelion. This result is exactly the same as equation (4.5.5) in Ref. [5]. Although the quantity (32) is small, the effect is cumulative, and eventually becomes susceptible to observation. It is greatest for the planet Mercury, which is the one closest to the Sun, and amounts to 43″ per century. There is excellent agreement between the theoretical and observed values.

Appendix

$$\cos\frac{\theta}{3} - \cos\left(\frac{\theta}{3}+\frac{2\pi}{3}\right) = -2\sin\left(\frac{\theta}{3}+\frac{\pi}{3}\right)\sin\left(-\frac{\pi}{3}\right) = 2\sin\frac{\pi}{3}\sin\left(\frac{\theta}{3}+\frac{\pi}{3}\right),$$

when $\theta\in[0,\pi]$, $\frac{\theta}{3}+\frac{\pi}{3}\in[\frac{\pi}{3},\frac{2\pi}{3}]$, $\sin(\frac{\theta}{3}+\frac{\pi}{3})\geq 0$. Therefore

$$\cos\frac{\theta}{3} \geq \cos\left(\theta+\frac{2\pi}{3}\right). \tag{33}$$

In the same way, we can obtain

$$\cos\frac{\theta}{3} - \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = -2\sin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right)\sin\left(-\frac{2\pi}{3}\right)$$
$$= 2\sin\frac{2\pi}{3}\sin\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \geq 0.$$

Therefore

$$\cos\frac{\theta}{3} \geq \cos\left(\theta + \frac{4\pi}{3}\right), \tag{34}$$

$$\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) - \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) = -2\sin(\frac{\theta}{3} + \pi)\sin\left(-\frac{\pi}{3}\right)$$
$$= 2\sin\frac{\pi}{3}\sin\left(\frac{\theta}{3} + \pi\right) \leq 0.$$

Therefore

$$\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \leq \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right). \tag{35}$$

From (33), (34) and (35), we know

$$\cos\left(\frac{\theta}{3} + \frac{2\pi}{3}\right) \leq \cos\left(\frac{\theta}{3} + \frac{4\pi}{3}\right) \leq \cos\frac{\theta}{3}.$$

It is very easy to see that

$$\left(-\frac{4p}{3}\right)^{\frac{1}{2}} > 0.$$

So $u_1 \leq u_2 \leq u_3$.

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